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Inverse Relations and Functions

Unit 1 Lesson 7

Students will be able to:

Find inverse functions.

Solve an equation of the form $f(x) = \mathrm{c}$ for a simple function f that has an inverse and write an expression for the inverse.

Verify by composition that one function is the inverse of another.

Read values of an inverse function from a graph or a table, given that the function has an inverse.

Produce an invertible function from a non-invertible function by restricting the domain.

Key Vocabulary:

Inverse function

One-to one function

Horizontal line test

The inverse of a relation is a relation obtained by reversing or swapping the coordinates of each ordered pair in the relation.

If the relation is described by an equation in the variables $\boldsymbol{x}$ and $\boldsymbol{y}$, the equation of the inverse relation is obtained by replacing every $\boldsymbol{x}$ in the equation with $\boldsymbol{y}$ and every $\boldsymbol{y}$ in the equation with $\boldsymbol{x}$.

If $f(x)$ represents a function of x, the inverse of the function is represented by the symbol $f^{-1}(x)$.

$$f^{-1}(x) \neq \frac{1}{f(x)}.$$

Sample Problem 1: **Find the inverse of each relation given as a set of ordered pairs.**

a.

x	0	1	2	4
y	-2	-3	-5	-1

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b.

x	-2	-5	-6	1
y	-6	-10	-3	-9

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x	-2	-5	-6	1
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y	-2	-5	-6	1

Horizontal Line Test

A function $\boldsymbol{f}$ has an inverse function $\boldsymbol{f^{-1}}$ if and only if each horizontal line intersects the graph of the function in at most one point.

If a function passes the horizontal line test, then it is said to be one-to-one, because no $\boldsymbol{x}$-value is matched with more than one $\boldsymbol{y}$-value and no $\boldsymbol{y}$-value is matched with more than one $\boldsymbol{x}$ -value.

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Horizontal Line Test

If a function $\boldsymbol{f}$ is one-to-one, it has an inverse function $\boldsymbol{f^{-1}}$ such that the domain of $\boldsymbol{f}$ is equal to the range of $\boldsymbol{f^{-1}}$, and the range of $\boldsymbol{f}$ is equal to the domain of $\boldsymbol{f^{-1}}$.

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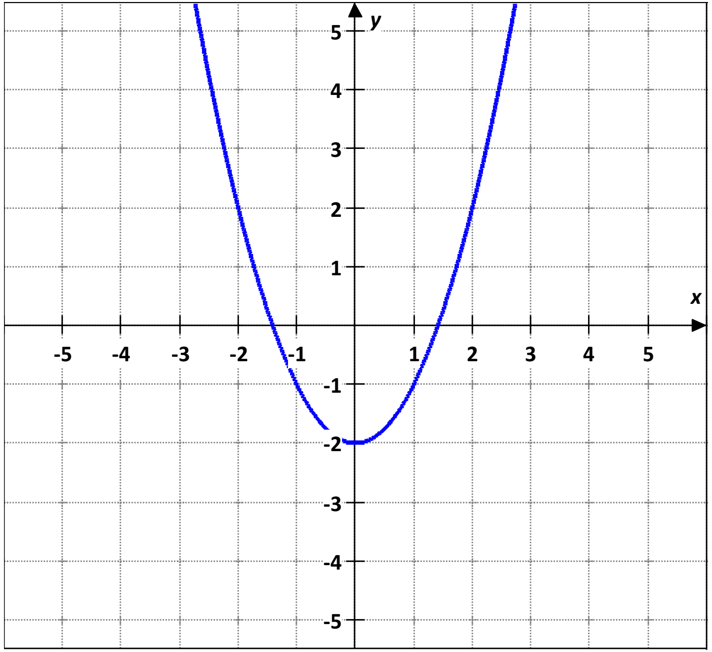
Sample Problem 2: **Use a horizontal line test to determine whether of the graph of each function is a one-to-one function.**

a. $f(x) = x^2 - 2$



Sample Problem 2: Use a horizontal line test to determine whether of the graph of each function is a one-to-one function.

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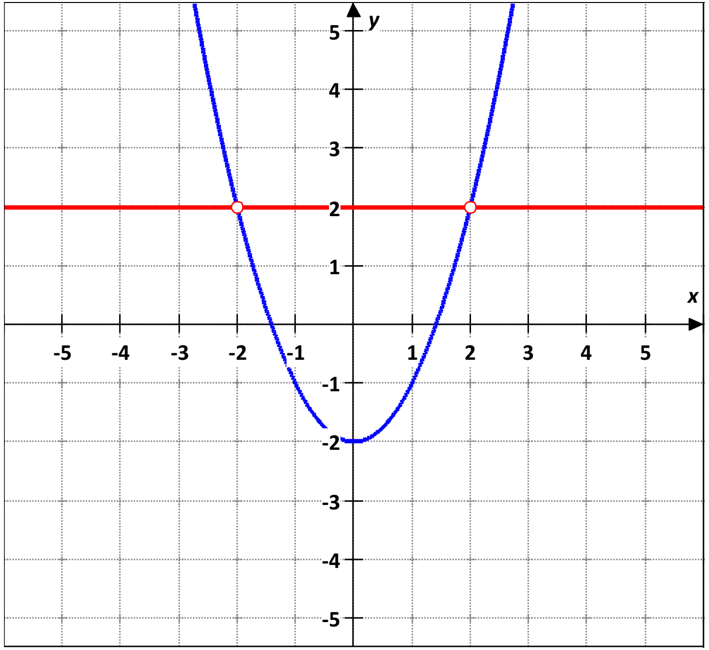
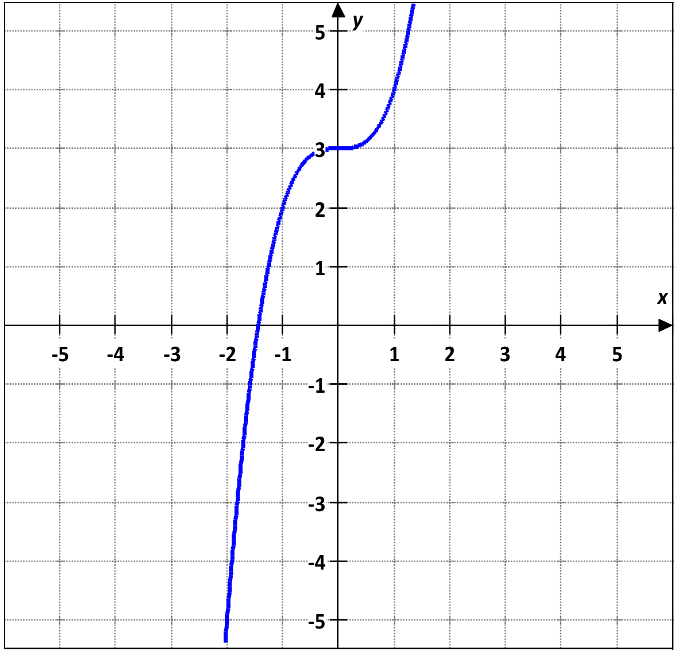


Because we can draw at least one horizontal line that intersects the graph more than once,
$f(x) = x^2 - 2$ is not a one-to-one function.
Therefore the inverse of $f(x) = x^2 - 2$ is a relation, but not a function.

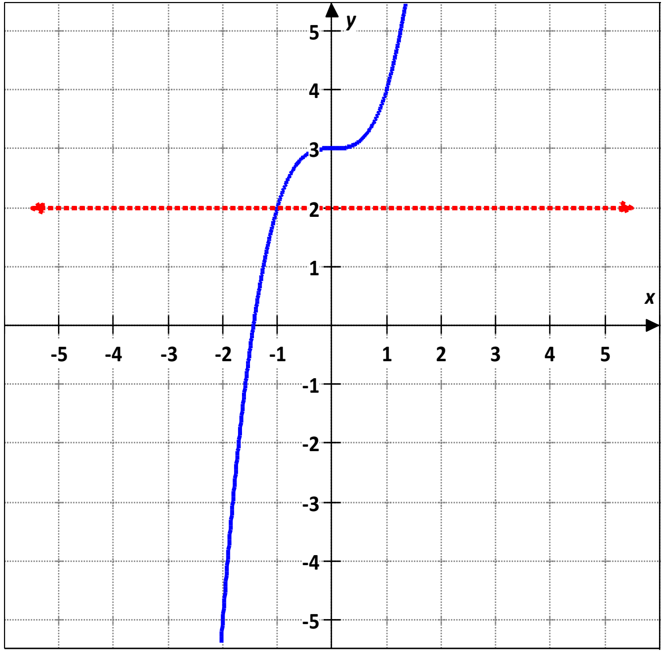
Sample Problem 2: **Use a horizontal line test to determine whether of the graph of each function is a one-to-one function.**

b. $f(x) = x^3 + 3$



Sample Problem 2: Use a horizontal line test to determine whether of the graph of each function is a one-to-one function.

b. $f(x) = x^3 + 3$



Because every horizontal line we draw intersects the graph only once, $f(x) = x^3 + 3$ is a one-to-one function. Therefore the inverse of $f(x) = x^3 + 3$ is a function.

Step by Step Procedure to Find the Inverse of $f(x)$

1. Determine whether the function has an inverse by checking to see if it is one-to-one.
2. Replace $f(x)$ with y.
3. Interchange x and y.
4. Solve for y.

Step by Step Procedure to Find the Inverse of $f(x)$

5. Replace $\boldsymbol{y}$ with $\boldsymbol{f^{-1}(x)}$.

6. State any restrictions on the domain of $\boldsymbol{f^{-1}(x)}$. Sometimes only part of the function you find algebraically may be the inverse function of $\boldsymbol{f}$. Therefore, be sure to analyze the domain of $\boldsymbol{f}$ when finding $\boldsymbol{f^{-1}}$.

The graphs of the function and the inverse function are reflections across the line $\boldsymbol{y = x}$.

Sample Problem 3: Find the inverse function, state any restrictions on its domain and then graph the function and its inverse.

a. $f(x) = 4x - 3$

Sample Problem 3: **Find the inverse function, state any restrictions on its domain and then graph the function and its inverse.**

a. $f(x) = 4x - 3$ $\quad D = (-\infty, \infty) \quad R = (-\infty, \infty)$

$$y = 4x - 3$$

$$x = 4y - 3$$

$$x + 3 = 4y - 3 + 3$$

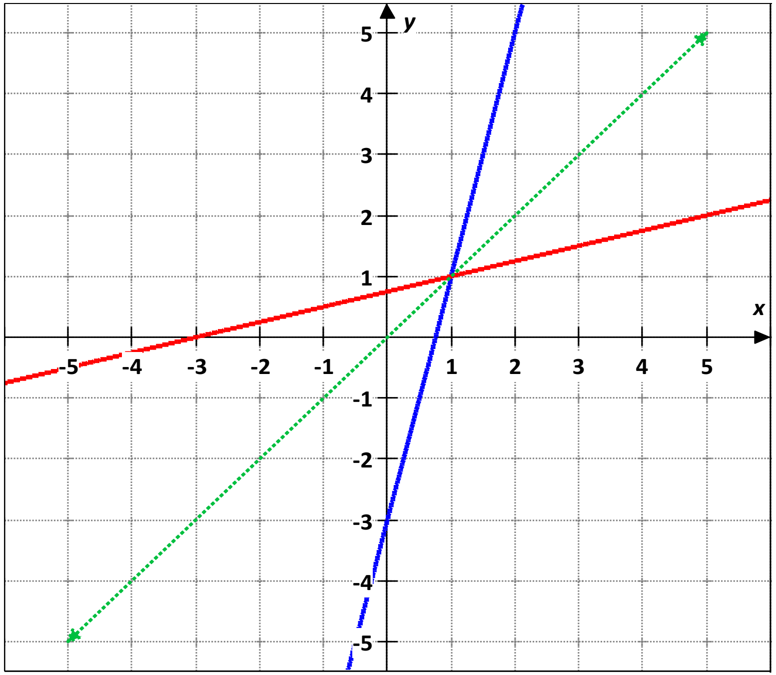
$$x + 3 = 4y$$

$$\frac{x+3}{4} = y$$

$$f^{-1}(x) = \frac{x+3}{4} \quad D = (-\infty, \infty) \quad R = (-\infty, \infty)$$

Sample Problem 3: **Find the inverse function, state any restrictions on its domain and then graph the function and its inverse.**

a. $f(x) = 4x - 3$ → $f^{-1}(x) = \frac{x+3}{4}$ →



Sample Problem 3: Find the inverse function, state any restrictions on its domain and then graph the function and its inverse.

b. $f(x) = (x-2)^3$

Sample Problem 3: **Find the inverse function, state any restrictions on its domain and then graph the function and its inverse.**

b. $f(x) = (x-2)^3$ $D = (-\infty, \infty)$ $R = (-\infty, \infty)$

$$y = (x-2)^3$$

$$x = (y-2)^3$$

$$\sqrt[3]{x} = \sqrt[3]{(y-2)^3}$$

$$\sqrt[3]{x} = y - 2$$

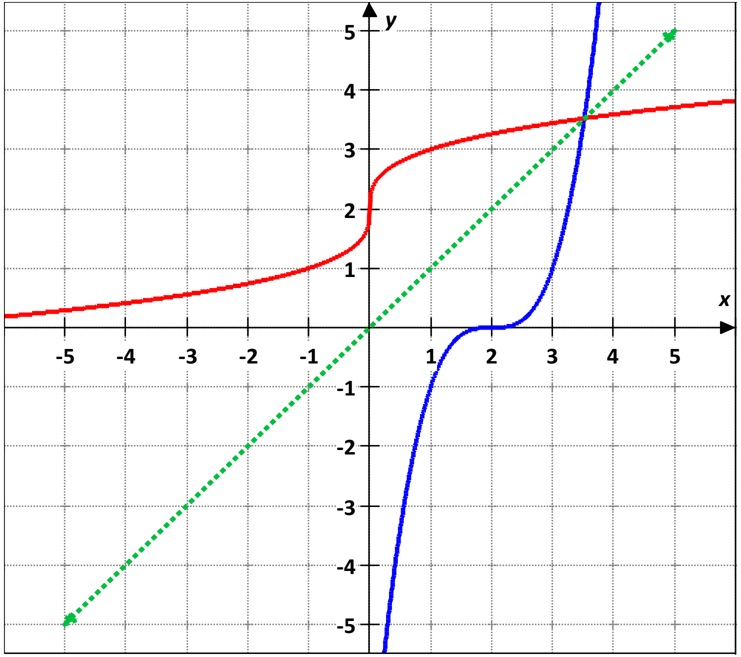
$$\sqrt[3]{x} + 2 = y - 2 + 2$$

$$\sqrt[3]{x} + 2 = y$$ $D = (-\infty, \infty)$ $R = (-\infty, \infty)$

$$f^{-1}(x) = \sqrt[3]{x} + 2$$

Sample Problem 3: **Find the inverse function, state any restrictions on its domain and then graph the function and its inverse.**

b. $f(x) = (x-2)^3$ $\longrightarrow$ $f^{-1}(x) = \sqrt[3]{x} + 2$ $\longrightarrow$



Sample Problem 3: Find the inverse function, state any restrictions on its domain and then graph the function and its inverse.

c. $f(x) = 2\sqrt{x+1}$

Sample Problem 3: Find the inverse function, state any restrictions on its domain and then graph the function and its inverse.

c. $f(x) = 2\sqrt{x+1}$ $\quad D = [-1, \infty)$ $\quad R = [0, \infty)$

$$y = 2\sqrt{x+1}$$

$$x = 2\sqrt{y+1}$$

$$x^2 = 2^2\left(\sqrt{y+1}\right)^2$$

$$x^2 = 4(y+1)$$

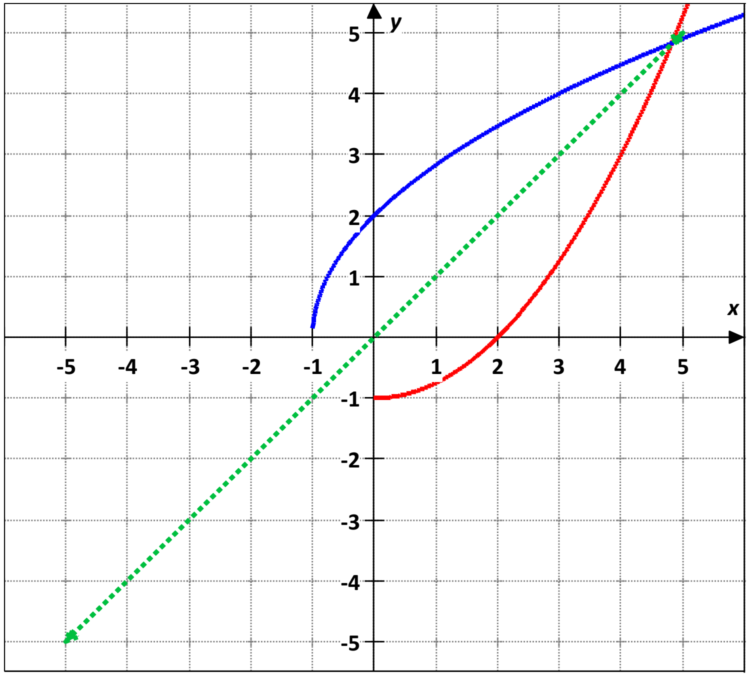
$$x^2 = 4y + 4$$

$$\frac{x^2 - 4}{4} = y \qquad f^{-1}(x) = \frac{x^2-4}{4} \quad x \geq 0$$

$D = [0, \infty)$ $\quad R = [-1, \infty)$

Sample Problem 3: **Find the inverse function, state any restrictions on its domain and then graph the function and its inverse.**

c. $f(x) = 2\sqrt{x+1}$ → $f^{-1}(x) = \dfrac{x^2 - 4}{4} \quad x \geq 0$ →



Compositions of Inverse Functions

Two functions, $\boldsymbol{f}$ and $\boldsymbol{g}$, are inverse functions if and only if:

- $\boldsymbol{f(g(x)) = x}$, for every $\boldsymbol{x}$ in the domain of $\boldsymbol{g(x)}$ and
- $\boldsymbol{g(f(x)) = x}$, for every $\boldsymbol{x}$ in the domain of $\boldsymbol{f(x)}$.

Sample Problem 4: **Show algebraically that f and g are inverse functions.**

a. $f(x) = 7x - 4$ $\quad\quad\quad$ $g(x) = \dfrac{x+4}{7}$

Sample Problem 4: **Show algebraically that f and g are inverse functions.**

a. $f(x) = 7x - 4$ $\qquad g(x) = \dfrac{x+4}{7}$

$$f(g(x)) = 7(g(x)) - 4$$

$$f(g(x)) = 7\left(\frac{x+4}{7}\right) - 4$$

$$f(g(x)) = x + 4 - 4$$

$$f(g(x)) = x$$

Sample Problem 4: **Show algebraically that f and g are inverse functions.**

a. $f(x) = 7x - 4$ $\quad\quad g(x) = \dfrac{x+4}{7}$

$$g(f(x)) = \frac{f(x)+4}{7}$$

$$g(f(x)) = \frac{7x-4+4}{7}$$

$$g(f(x)) = \frac{7x}{7}$$

$$g(f(x)) = x$$

Sample Problem 4: **Show algebraically that f and g are inverse functions.**

b. $f(x) = \sqrt{x+5} - 3$ $\qquad g(x) = x^2 + 6x + 4 \qquad x \geq 3$

Sample Problem 4: Show algebraically that f and g are inverse functions.

b. $f(x) = \sqrt{x+5} - 3$ $\quad g(x) = x^2 + 6x + 4 \quad x \geq 3$

$$f(g(x)) = \sqrt{g(x) + 5} - 3$$

$$f(g(x)) = \sqrt{x^2 + 6x + 4 + 5} - 3$$

$$f(g(x)) = \sqrt{x^2 + 6x + 9} - 3$$

$$f(g(x)) = \sqrt{(x+3)^2} - 3$$

$$f(g(x)) = (x+3) - 3$$

$$f(g(x)) = x$$

Sample Problem 4: Show algebraically that f and g are inverse functions.

b. $f(x) = \sqrt{x+5} - 3$ $\quad g(x) = x^2 + 6x + 4 \quad x \geq 3$

$$g(f(x)) = (f(x))^2 + 6 * f(x) + 4$$

$$g(f(x)) = (\sqrt{x+5} - 3)^2 + 6 * (\sqrt{x+5} - 3) + 4$$

$$g(f(x)) = x + 5 - 6\sqrt{x+5} + 9 + 6\sqrt{x+5} - 18 + 4$$

$$g(f(x)) = x$$