The inverse of a relation is a relation obtained by reversing or swapping the coordinates of each ordered pair in the relation.

If the relation is described by an equation in the variables $x$ and $y,$ the equation of the inverse relation is obtained by replacing every $x$ in the equation with $y$ and every $y$ in the equation with $x$.

If $f\left(x\right) $represents a function of$ x$, the inverse of the function is represented by the symbol $f^{-1}\left(x\right)$. $f^{-1}\left(x\right)\ne \frac{1}{f\left(x\right)}$**.**

**Sample Problem 1: Find the inverse of each relation given as a set of ordered pairs.**

|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| **a.**  |

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| $$x$$ | $$0$$ | $$1$$ | $$2$$ | $$4$$ |
| $$y$$ | $$-2$$ | $$-3$$ | $$-5$$ | $$-1$$ |

 |

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| $$x$$ | $$-2$$ | $$-3$$ | $$-5$$ | $$-1$$ |
| $$y$$ | $$0$$ | $$1$$ | $$2$$ | $$4$$ |

 |
| **b.**  |

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| $$x$$ | $$-2$$ | $$-5$$ | $$-6$$ | $$1$$ |
| $$y$$ | $$-6$$ | $$-10$$ | $$-3$$ | $$-9$$ |

 |

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| $$x$$ | $$-6$$ | $$-10$$ | $$-3$$ | $$-9$$ |
| $$y$$ | $$-2$$ | $$-5$$ | $$-6$$ | $$1$$ |

 |

**Horizontal Line Test**

A function$ f$ has an inverse function $f^{-1}$ if and only if each horizontal line intersects the graph of the function in at most one point.If a function passes the horizontal line test, then it is said to be one-to-one, because no $x$-value is matched with more than one $y$-value and no $y$-value is matched with more than one $x$ -value.

If a function$ f$ is one-to-one, it has an inverse function $f^{-1}$ such that the domain of$ f$ is equal to the range of$ f^{-1}$, and the range of$ f$ is equal to the domain of$ f^{-1}$.

**Sample Problem 2: Use a horizontal line test to determine whether of the graph of each function is a one-to-one function.**

|  |  |  |
| --- | --- | --- |
| **a.**  | $$f\left(x\right)=x^{2}-2$$ | $$f\left(x\right)=x^{2}-2$$ |
|  | Because we can draw at least one horizontal line that intersects the graph more than once, $f\left(x\right)=x^{2}-2$ is not a one-to-one function. Therefore the inverse of$ f\left(x\right)=x^{2}-2$ is a relation, but not a function. |
| **b.**  | $$f\left(x\right)=x^{3}+3$$ | $$f\left(x\right)=x^{3}+3$$ |
|  | Because every horizontal line we draw intersects the graph only once, $f\left(x\right)=x^{3}+3$ is a one-to-one function. Therefore the inverse of$ f\left(x\right)=x^{3}+3$ is a function. |

**Step by Step Procedure to Find the Inverse of** $f\left(x\right)$

1. Determine whether the function has an inverse by checking to see if it is one-to-one.
2. Replace $f\left(x\right) $with $y.$
3. Interchange $x$ and $y.$
4. Solve for$ y.$
5. Replace $y$ with $f^{-1}\left(x\right).$
6. State any restrictions on the domain of $f^{-1}\left(x\right)$. Sometimes only part of the function you find algebraically may be the inverse function of$ f$. Therefore, be sure to analyze the domain of$ f$ when finding$ f^{-1}.$

The graphs of the function and the inverse function are reflections across the line $y=x.$

**Sample Problem 3: Find the inverse function, state any restrictions on its domain and then graph the function and its inverse.**

|  |  |  |
| --- | --- | --- |
| **a.**  | $f\left(x\right)=4x-3 $ |  |
|  | $f\left(x\right)=4x-3 $$ D=(-\infty ,\infty ) R= (-\infty ,\infty )$$$y=4x-3$$$$x=4y-3$$$$x+3=4y-3+3$$$$x+3=4y$$$$\frac{x+3}{4}=y f^{-1}\left(x\right)=\frac{x+3}{4} $$$ D=(-\infty ,\infty ) R= (-\infty ,\infty )$$f\left(x\right)=4x-3 $$$f^{-1}\left(x\right)=\frac{x+3}{4}$$ |  |
| **b.**  | $$f\left(x\right)=\left(x-2\right)^{3}$$ |  |
|  | $f\left(x\right)=\left(x-2\right)^{3} $$ D=(-\infty ,\infty ) R= (-\infty ,\infty )$$$y=\left(x-2\right)^{3}$$$$x=\left(y-2\right)^{3}$$$$\sqrt[3]{x}=\sqrt[3]{\left(y-2\right)^{3}} $$$$\sqrt[3]{x}=y-2 $$$$\sqrt[3]{x}+2=y-2+2$$$$\sqrt[3]{x}+2=y$$$f^{-1}\left(x\right)=\sqrt[3]{x}+2$$ D=(-\infty ,\infty ) R= (-\infty ,\infty )$ | $ f\left(x\right)=\left(x-2\right)^{3} $$$f^{-1}\left(x\right)=\sqrt[3]{x}+2$$ |
| **c.**  | $$f\left(x\right)=2\sqrt{x+1}$$ |  |
|  | $$f\left(x\right)=2\sqrt{x+1} D=\left[-1,\infty \right) R= \left[0,\infty \right) $$$$y=2\sqrt{x+1}$$$$x=2\sqrt{y+1}$$$$x^{2}=2^{2}\left(\sqrt{y+1}\right)^{2} $$$$x^{2}=4\left(y+1\right) $$$$x^{2}=4y+4 $$$$x^{2}-4=4y+4-4 $$$$x^{2}-4=4y$$$$\frac{x^{2}-4}{4}=y $$$$f^{-1}\left(x\right)=\frac{x^{2}-4}{4} x\geq 0$$$$D=\left[0,\infty \right) R= \left[-1,\infty \right)$$$ $ | $f\left(x\right)=2\sqrt{x+1} $$$f^{-1}\left(x\right)=\frac{x^{2}-4}{4} x\geq 0$$ |

**Compositions of Inverse Functions**

Two functions, $f$ and$ g$, are inverse functions if and only if:

* $f\left(g(x)\right)=x, $for every $x$ in the domain of $g(x)$ and
* $g\left(f(x)\right)=x,$ for every $x$ in the domain of $f\left(x\right)$.

**Sample Problem 4:**  **Show algebraically that** $f$ **and**$ g$ **are inverse functions.**

|  |  |  |  |
| --- | --- | --- | --- |
| **a.**  | $$f\left(x\right)=7x-4$$$$g\left(x\right)=\frac{x+4}{7}$$ | **b.** | $$f\left(x\right)=\sqrt{x+5}-3 $$$$g\left(x\right)=x^{2}+6x+4 x\geq 3$$ |
|  | $$f\left(g\left(x\right)\right)=7\left(g\left(x\right)\right)-4 $$$$f\left(g\left(x\right)\right)=7\left(\frac{x+4}{7}\right)-4 $$$$f\left(g\left(x\right)\right)=x+4-4 $$$$f\left(g\left(x\right)\right)=x $$$$g\left(f\left(x\right)\right)=\frac{f\left(x\right)+4}{7}$$$$g\left(f\left(x\right)\right)=\frac{7x-4+4}{7}$$$$g\left(f\left(x\right)\right)=\frac{7x}{7}$$$$g\left(f\left(x\right)\right)=x$$ |  | $$f\left(g\left(x\right)\right)=\sqrt{g\left(x\right)+5}-3 $$$$f\left(g\left(x\right)\right)=\sqrt{x^{2}+6x+4+5}-3$$$$f\left(g\left(x\right)\right)=\sqrt{x^{2}+6x+9}-3$$$$f\left(g\left(x\right)\right)=\sqrt{\left(x+3\right)^{2}}-3$$$$f\left(g\left(x\right)\right)=\left(x+3\right)-3$$$$f\left(g\left(x\right)\right)=x$$$$g\left(f\left(x\right)\right)=\left(f\left(x\right)\right)^{2}+6\*f\left(x\right)+4$$$$g\left(f\left(x\right)\right)=\left(\sqrt{x+5}-3\right)^{2}+6\*\left(\sqrt{x+5}-3\right)+4$$$$g\left(f\left(x\right)\right)=x+5-6\sqrt{x+5}+9+6\sqrt{x+5}-18+4$$$$g\left(f\left(x\right)\right)=x$$ |